**NUMERICAL METHODS
FOR MATHEMATICAL PHYSICS INVERSE PROBLEMS**

## Lecture 7. Well-posedness of the optimization problems

We know that the inverse problems can be transformed to the problems of finding of extremum. So the practical methods of inverse problems theory are based on the optimization methods. The problems of the minimization of the functionals can be solved by means of the gradient methods. We applied it for the cases with direct dependence of the functional from unknown parameter of the system. However for the standard optimization control problems this dependence is not direct. In really the functional depends from the state function; and the state function depends from unknown parameter (control) by the state equation. It is true for optimization control problems, which are the transformation of mathematical physics inverse problems. So we will try to extend the known minimization methods to the optimization control problems.

### 7.1. Ill-posedness in the sense of Tikhonov

So far we have been dealing with extremum problems with quite unfavourable properties. In particular, there were cases where optimal controls did not exist or were nonunique. In contrast to this, the main subject of the present example is an optimal control problem that has a unique solu­tion. In some of the previous examples, the optimality conditions were not sufficient. In the present example, the optimality conditions in the form of the maximum principle are both necessary and sufficient. Another problem with the optimality conditions was that they were either unsolvable or had too many solutions. In the following example, there is a unique optimal control that satisfies the maximum condition. It may seem that no unexpected difficulties can be encountered.

Nevertheless, surprises are not over yet. We will show that in the present example it is possible to construct a sequence of admissible controls such that the values of the functional at the elements of this sequence converge to its minimum value, although the sequence itself does not converge to an optimal control. This means that finding a solution of the extremum problem with the desired accuracy is not guaranteed even under such favourable conditions.

The described situation is characteristic for optimization problems which are not well-posed in the sense of Tikhonov. In what follows, we present sufficient conditions for the problem to be well-posed. We also describe the regularization methods for ill-posed optimal control problems that allow us to overcome various difficulties in some situations.

Let the state of the system be described by the Cauchy problem

 (5.1)

The control *и= u(t)* is assumed to belong to the set

*U =* { *u*∈*L*2(0,1)| | *u*(*t*) | ≤ 1 , *t*∈(0,1) } .

The optimality criterion is represented by the integral functional



**Problem 5**. Find a control that minimizes the functional *I* on *U.*

Although this problem was already considered when we were studying singular controls (see Problem 2*"*), we will show that not all of its issues have been clarified.

 The solvability of Problem 5 can be easily established using Theorem 5. Indeed, we already established all the necessary properties of the functional to be minimized and the set of admissible controls during the analysis of similar problems. Taking into account the strict convexity of the functional, which can be proved the same way as in Problem 0 (see Example 1). We can also establish the uniqueness of the optimal control using Theorem 2. Besides, similar results can be obtained by analyzing the optimality conditions.

We showed in Example 2 that the maximum principle for Problem 5 is defined by the formula

 (5.2)

where *p* is a solution of the problem

 (5.3)

As we know, the system of optimality conditions (5.1)-(5.3) does not have nonsingular solutions. The maximum principle can only hold if it is degenerate, which corresponds to the case *p=*0. From (5.3) it follows that *x=*0. Finally, using (5.1), we find the unique solution of the maximum principle — the singular control *u*0= 0.

**Conclusion 5.1.** *The maximum principle for Problem 5 has a unique solution — the singular control u*0*.*

At the same time, the functional to be minimized is nonnegative. It vanishes only for *x=*0, i.e., at the control *u*0*.* We thus obtain very positive results.

**Conclusion 5.2.** Problem 5 has a unique solution *u*0.

**Conclusion 5.3.** *The maximum principle for Problem 5 is a necessary and sufficient; optimality condition.*

**Remark 5.1.** The sufficiency of the maximum principle can be estab­lished directly by applying Theorem 3, as was done in Example 1.

It may seem that there is no reason to return to such a simple problem that has been analyzed sufficiently well. However, we will show that this problem has more surprises than we expect.

**Remark 5.2.** The existence of a singular control is already an important sign of upcoming difficulties.

 Consider the sequence of controls defined by the following formulas (see Figure 26):

*uk*(*t*) = sin *kt* , *k* = 1,2, … . (5.4)

These functions are infinitely differentiable and are not greater than unity in absolute value. Hence, these controls are admissible.



Figure 26. The minimizing sequence in Problem 5.

The corresponding solutions of problem (5.1) are defined as follows (see Figure 27):

 (5.5)



Figure 27. The sequence of states defined by formula (5.5)

The following estimate holds:



Hence,



Then the sequence of functional corresponding to the controls *uk* tends to zero, i.e., the minimum value of the optimality criterion on the set of admissible controls.

**Conclusion 5.4.** *The sequence* {*uk*} *in Problem 5 is minimizing.*

Under these conditions, it may seem natural to take a function *uk* for sufficiently large *k* as an approximation of the optimal control. The question arises of whether the minimizing sequence {*u*k} converges to the optimal control *u*0. If this is true, the norm of the difference (*u*k-*u*0) must tend to zero. We now estimate this norm:

 **

Thus, the optimal control *u*0 is not the limit of the minimizing sequence {*u*k}.

**Conclusion 5.5.** *The minimizing sequence* {*u*k} *does not converge to the optimal control.*

**Remark 5.3**. The above conclusion is rather obvious. It suffices to compare the elements *u*k(sinusoids with indefinitely increasing oscillation frequency) with the optimal control identically equal to zero. See Figure 26.

The sequence {*u*k}does not converge in any conventional functional space. This is not surprising since we already encountered the same problem with nonconvergent minimizing sequence in two previous examples. How­ever, there was no optimal control in those examples, i.e., there was no possible limit for the minimizing sequence. Since the existence of an op­timal control in the present example is obvious, we arrive at the following unfortunate conclusion.

**Conclusion 5.6.** *Not every minimizing sequence converges to the opti­mal control in the present example.*

Summing up these results, we can subdivide all optimal control problems into two classes. The problem of optimal control is called *well-posed in the sense of Tikhonov* if every minimizing sequence for this problem converges to the optimal control. If there exits a minimizing sequence that does not converge to the optimal control, then the problem is called *ill-posed in the sense of Tikhonov.*

**Conclusion 5.7.** *Problem 5 is ill-posed in the sense of Tikhonov,*

Thus, even though there is a unique optimal control and the optimality conditions are necessary and sufficient, easy solution is not guaranteed. In ill-posed problems, even if we are able to find a control at which the value of the functional being minimized is as close to its lower bound as desired, this control is not guaranteed to be close enough to the optimal control.

**Remark 5.4.** The minimizing sequence defined above weakly converges to the optimal control. For this reason, the problem could be called weakly well-posed in the sense of Tikhonov. However, this positive result is not re­ally meaningful because the elements of the minimizing sequence (frequently oscillating functions) are by no means close to the optimal control, which is constant.

**Remark 5.5.** Note that the weak convergence of the minimizing se­quence is of no surprise. The Banach—Alaoglu theorem is still applicable here since the set of admissible controls is bounded.

We now try to explain why the problem in question is not well-posed in the sense of Tikhonov. Apparently, in problems of this kind, the functional is not very sensitive to the variations of the control. Under this condition, a substantial variation of the control has only a weak effect on the optimality criterion. Therefore, the value of the functional at a control considerably different from the optimal one may be relatively close to its minimum value.

**Conclusion 5.8.** *The problem is ill-posed in the sense of Tikhonov because the optimality criterion is not sensitive enough to the variation of the control.*

The question arises of whether the fact that the problem is not well-posed in the sense of Tikhonov is a really serious obstacle. In applied problems, we always seek approximate solutions. Therefore, it may be reasonable to always seek the approximate solution in the form of an admissible control at which the value of the functional is sufficiently close to its lower bound. With such an approach, the fact that a certain problem is not well-posed seems to be of no concern.

Suppose that we have a situation where the present example has physical meaning. As we know, the exact optimal control is the function identically equal to zero, which is very simple and has no problem being represented in practice. At the same time, when we seek an approximate solution, it turns out to be a sinusoid with high frequency of oscillations. Although the corresponding value of the functional is sufficiently small, the obtained con­trol is not satisfactory as far as practical application is concerned. For this reason, determining optimal controls with the required accuracy is a more preferable way, though finding the approximate minimum of the functional may also be satisfactory enough in some cases.

**Remark 5.6.** If a problem is ill-posed in the sense of Tikhonov, numer­ical algorithms are usually very sensitive to different kinds of errors. This brings up the notion of ill-posedness in the sense of Hadamard, which is the subject of the next example.

The question arises of *whether a minimizing sequence that does not con­verge to the optimal control may converge to any other limit.* If the functional is continuous, the convergence of a sequence of controls obviously implies the convergence of the corresponding sequence of the functional values. So if the sequence of controls converges to a limit which is not an optimal control, then the corresponding sequence of the functional values will converge to the value of the functional at this limit. In this case, however, the sequence is not minimizing since the values of the functional at this sequence do not converge to its lower bound.

**Conclusion 5.9.** *The minimizing sequence either converges to the op­timal control or does not converge at all.*

It is interesting to know how common is the case where the minimizing sequence in an ill-posed problem does not converge. In the present example, it is easy to see every sequence of admissible controls that weakly converges to zero is minimizing. Since the class of weakly converging sequences is substantially larger than that of strongly converging sequences, we conclude that minimizing sequences usually do not converge in problems that are ill-posed in the sense of Tikhonov.

**Conclusion 5.10.** *Minimizing sequences in ill-posed problems usually do not converge.*

In unsolvable optimization problems, minimizing sequences obviously do not converge to optimal controls since they don't exist.

**Conclusion 5.11.** Unsolvable optimization *problems are* *ill-posed in the sense of Tikhonov*.

This brings up the question: *I*s *it possible for an optimal control prob­lem with more than one solution to be well-posed in the sense of Tikhonov?* Consider the case where the optimal control problem has two different so­lutions *и* and *v.* Let {*uk*}and {*vk*}be sequences of admissible controls converging to *u* and *v,* respectively. If the functional *I* to be minimized is continuous, we have

.

Let {*wk*} be a sequence whose elements with odd indices are the elements of {*uk*} and those with even indices are the elements of {*v*k}*.* We have *I*(*w*k)→inf *I*(*U*)because the same is true for all subsequences of {*I*(*wk*)}. Hence, {*wk*}is a minimizing sequence. At the same time, it does not con­verge because two of its subsequences converge to different limits. It follows that this problem is ill-posed in the sense of Tikhonov.

**Conclusion 5.12.** *Optimization problems with more than one solution are ill-posed in the sense of Tikhonov.*

We see that the class of optimal control problems well-posed in the sense of Tikhonov is smaller than the class of problems that have a unique solution.

In particular, Problem 5 is uniquely solvable, but it is ill-posed in the sense of Tikhonov. For this reason, in order to prove that a problem is well-posed, the restrictions to be imposed on the system must be stronger than those we had when establishing the existence and uniqueness of an optimal control. In what follows, we find the conditions that guarantee the well-posedness of the optimal control problem in the sense of Tikhonov.

 In the proof of the solvability of the extremum problem, we have used the convexity of the functional to be minimized. To provide the uniqueness of the optimal control, we needed the stronger property of strict convexity. To establish that the optimal control problem is well-posed in the sense of Tikhonov, we need an even stronger condition — the strict uniform convexity of the functional.

A functional *I* defined on a convex set *U* is called *strictly uniformly convex* if there exists a continuous function *δ=δ*(*τ*)such that







and for every two elements *u,vU* and *α*[0,1] the following inequality elements holds:



Obviously, every strictly uniformly convex functional is strictly convex.

Suppose that a function *и* is a solution of the problem of minimizing a strictly uniformly convex functional *I* on a convex set *U.* Consider an arbitrary minimizing sequence, i.e., a sequence of *ukU* such that *I*(*uk*)→inf *I*(*U*)*.* Then



and therefore



Since the control *и* is optimal, the left-hand side of the foregoing inequality is nonnegative. As a result, we have



We now pass to the limit as *α*→0 since the parameter *α* is arbitrary. This yields the inequality



Since {*uk*}is a minimizing sequence, we have



Assume that the sequence of positive numbers {*τk*}, where *,* does not tend to zero. This means that it either does not converge or tends to a positive number. From the properties of the function *δ,* it follows that {*δ*(*τk*)} either converges to a nonzero limit or does not converge at all. In any case, it does not converge to zero, which leads to a contradiction. We have thus proved that *uk*→*u.* Therefore, every minimizing sequence converges to the optimal control, which means that this extremum problem is well-posed in the sense of Tikhonov.

**Conclusion 5.13**. *The strict uniform convexity of the functional is required to prove that the optimization problem is well-posed in the sense of Tikhonov.*

Taking into account Theorems 5 and 7 on the existence of a solution of the extremum problem, we arrive at the following conclusion.

**Theorem 8.** *The problem of minimizing a lower semicontinuous functional which is bounded from below and is strictly uniformly convex on a convex closed bounded subset of a Hilbert space is well-posed in the sense of Tikhonov. (The condition of boundedness for the subset can be replaced with the coerciveness of the functional.)*

This theorem will be used below to prove that the problem considered in Introduction is well-posed in the sense of Tikhonov.

 **Example.** Let the set

*U =* { *u* ∈ *L*2(0,1)| | *u*(*t*) | ≤ 1, *t*∈(0,1) } .

be the domain of definition for the functional



The system state *x* is described by the formulas

.

**Problem 5'**. Find a control  minimizing the functional *I* on *U*.

We are dealing with Problem 0, for which the existence and uniqueness of an optimal control was established earlier. All the assumptions of Theo­rem 8, except for the property of strict uniform convexity of the functional, were proved to hold.

In order to establish that the functional is strictly uniformly convex, we first consider the quadratic function



We have



for all  and all numbers *x, y.* Therefore,



It follows that *f* is strictly uniformly convex and .

Setting *x = u*(*t*) and *у* = *v*(*t*) integrating the foregoing equality with respect to *t,* we obtain



Hence, the quadratic functional *I* defined by the formula



is strictly uniformly convex, i.e.,

**

We already established the convexity of the functional



(seeExample 1), where *x*(*u*)is a solution of the Cauchy problem corre­sponding to the control *u.* Taking into account that *I* is the half sum of the functional *J* and *K,* we have



**Conclusion 5.14.** *The functional I is strictly uniformly convex.*

By Theorem 8, the problem is well-posed in the sense of Tikhonov.

**Conclusion 5*.*15***. Problem 5’ is well-posed in the sense of Tikhonov.*

In Problem 5, the functional to be minimized is strictly convex, but not strictly uniformly convex. Although this is enough to establish the uniqueness of the optimal control it is not enough for the problem to be well-posed.

**Conclusion 5*.*16***. Problem 5 is not well-posed in the sense of Tikhonov because the functional to be minimized is not strictly uniformly convex.*

 Numeric solution of ill-posed extremum problems may involve certain difficulties. In particular, minimization algorithms do not necessarily guarantee the obtaining of optimal controls with the desired accuracy. Various regularization methods can help to deal with these problems. For example, the Tikhonov regularization method for Problem 5 involves the use of the functional



where *ε* is the regularization parameter.

The problem of minimizing this functional coincides with Problem 0 up to a constant multiplier. As Problem 0 was proved to be well-posed, the solution of the regularized problem should not be difficult. After finding a solution we, we pass to the limit as . The initial calculations are made for sufficiently large *ε,* when the problem has good properties (numerical algorithms for this problem converge sufficiently well), although they are not as good as in Problem 5. The next stage of calculations is performed after *ε* is reduced, the initial approximation being the control from the previous step of the regularization method. The deterioration of convergence of the solution algorithm as the problem is becoming closer to the original ill-posed problem is partially compensated by the gradual refinement of the initial approximation of the control at every step of the regularization method. It is sometimes possible to solve the ill-posed problem using this algorithm.

**Remark 5.7.** The algorithm described above involves an imbedded iterative process.

### 7.2. Ill-posedness in the sense of Hadamard

In the previous example, we introduced the notion of well-posedness in the sense of Tikhonov for extremum problems, which means that every minimiz­ing sequence converges to the optimal control. There is an essentially dif­ferent notion of well-posedness in the theory of differential equations which implies the existence of a unique solution with continuous dependence on certain parameters. A similar property is also meaningful for optimization.

In what follows, we consider a uniquely solvable optimal control problem with optimality criterion containing a certain parameter. We will show that the dependence of the corresponding optimal control on this parameter is not continuous. This means that the problem is not well-posed in the sense of Hadamard. In this case, small errors in determining the parameters of the problem and various errors associated with the algorithm may lead to very substantial errors in the results of solution.

We reveal the relationship between the notions of well-posedness in the sense of Hadamard and well-posedness in the sense of Tikhonov. Using this relationship, it is possible to prove that an extremum problem is well-posed in the sense of Hadamard and perform the regularization of ill-posed problems. An example of the optimal control problem well-posed in the sense of Hadamard is presented.

 Let the state of the system be described by the Cauchy problem

 (6.1)

The control *и* = *u*(*t*) is assumed to belong to the set

*U =* { *u*∈*L*2(0,1)| | *u*(*t*) | ≤ 1 , *t*∈(0,1) }.

The optimality criterion is denned by the formula



where *yk*(*t*)*=*(*kπ*)*-*1sin*kπt* and *k* is a parameter.

**Problem 6.** Find a control *u*∈*U* minimizing the functional *Ik* on *U*.

**Remark 6.1.** For *x =* 0, this problem coincides with Problem 5 (up to the constant coefficient of the integral). In Problem 6, it is required to find an admissible control such that the system state at this control is as close to the given state *уk* as possible.

The specific feature of this problem is the presence of a parameter. In fact, it is not a single problem but a set of optimization problems. It is required not only to find the corresponding optimal control, but also to investigate the dependence of this control on the parameter *k.*

 We now solve Problem 6 directly. The functional to be minimized is non-negative. It vanishes only if *x*(*t*)*=yk*(t). Substituting this value into (6.1), we find the function



Since this control is admissible, we conclude that it is a unique solution of Problem 6.

**Conclusion 6.1.** *The problem of minimizing the functional Ik on U has a unique solution uk.*

**Remark 6.2.** It is easy to prove that the maximum principle for Prob­lem 6 is a necessary and sufficient optimally condition, the corresponding optimal control being singular.

Passing to the limit in the formula for the optimally criterion, we find the limit



The problem of minimizing the functional *I∞* on *U*, which almost coincides with Problem 5, has a unique solution *u∞ =* 0.

It is required to find out *whether the sequence of solutions* {*uk*} *converges to the solution u∞ of the limit problem.* We have



Hence, the control *u∞*, which is optimal for the limit problem, is not the limit of the sequence of solutions {*uk*}.

**Conclusion 6.2.** *The dependence of the solution of Problem 6 on the parameter к is not continuous.*

The obtained result represents another serious obstacle in the investiga­tion of extremum problems. The optimal control problem is called *well-posed in the sense of Hadamard* if it has a unique solution with continuous depen­dence on the problem parameters.

**Conclusion 6.3.** *Problem 6 is not well-posed in the sense of Hadamard.*

The ill-posedness of Problem 6 in the sense of Hadamard and the diffi­culties associated with this have the following implications. In the setting of any applied optimal control problem, as well as any other applied problem, there are parameters that are determined in the experiment and therefore contain errors. If the problem is well-posed in the sense of Hadamard, small errors in the system parameters do not cause substantial errors in the opti­mal control. Otherwise, we have a very different situation.

We know that the function *yk* will be as close to *y∞* = 0 as desired if *k* is sufficiently large. Assume that *y∞* is the true value (in the natural setting of the problem) corresponding to the functional *I∞*. Suppose that the value *yk* obtained in the measurements is sufficiently close to *y∞.* Then, instead of the true functional *I∞*,we have to minimize the functional *Ik* sufficiently close to *I∞.* Although we expect that small errors in the input data cause only a small error in the optimal control, we obtain a solution *uk* of Problem 6 which differs much from the true optimal control *u∞.* This is the way substantial errors are introduced in the solution results when solving ill-posed problems in the sense of Hadamard.

Negative effects appear when solving ill-posed problems in the sense of Hadamard even in the case of no errors in the problem parameters. This is caused by errors introduced at various stages of the calculation procedure (calculating special functions, square roots, rounding, etc.). Small calcula­tion errors result in large errors in the solution since the problem is ill-posed.

**Conclusion 6.4.** *The ill-posedness in the sense of Hadamard may cause serious difficulties in the practical solution of optimization problems.*

**Remark 6.3**. An extraordinary kind of ill-posedness in the sense of Hadamard will be demonstrated in Example 8: changing a parameter in the functional to be minimized causes a change in the number of solutions of the boundary value problem corresponding to the system of optimality conditions.

Our next purpose is to find out what kind of problems is well-posed in the sense of Hadamard and what to do if a problem is ill-posed.

 Although there is an essential difference between the notions of well-posedness in the sense of Tikhonov and well-posedness in the sense of Hadamard, they have much in common. In particular, Problem 6 (which is ill-posed in the sense of Hadamard) is ill-posed in the sense of Tikhonov for every *k.* For this reason, we may expect some similarity in the methods of establishing the well-posedness of both types, as well as in the methods of solution.

Consider the problem of minimizing a functional *Iμ* on the set *U,* where *μ* is a parameter with values in a set *M.*

**Theorem 9.** *Let the problem of minimizing the functional Iμ be well-posed in the sense of Tikhonov for every μ*∈*M and let the mapping μ→Iμ*(*v*) *be uniformly continuous on M with respect to v*∈*U. Then the extremum problem is well-posed in the sense of Hadamard.*

**Proof**. Consider an arbitrary sequence {*μk*}converging in *M.* There exists an element *μ*∈*M* such that *μk→μ.* For brevity, we denote the functional *Iμk* by *Ik* and the functional *Iμ* by *I.*

Every well-posed problem in the sense of Tikhonov has a unique solution. Let *uk* and *и* be the solutions of problems of minimizing the functionals *Ik* and *I* on *U,* respectively. It suffices to prove that *uk →* *и.* We have

*I*(*uk*)- *I*(*u*) *=* [*I*(*uk*)- *Ik*(*uk*)] *+* [*Ik*(*uk*) *- Ik*(*u*)]+ [*Ik*(*u*)- *I*(*u*)]*.*

Since *uk* and *и* are the solutions of the corresponding extremum problems, we obtain

0 ≤ *Ik*(*u*) – *I*(*u*) , *Ik*(*uk*) – *Ik*(*u*) ≤ 0 .

Hence,



The mapping *μ→Iμ* being uniformly continuous on *M* with respect *to v*∈*U* the expression in the right-hand side of this inequality tends to zero. Hence, *I*(*ик*) *→I*(*u*)*.* Therefore, {*uk*}is a minimizing sequence for the functional *I*=*Iμ.*

By assumption, the problem of minimizing the functional *Iμ* is well-posed in the sense of Tikhonov. Then the minimizing sequence {*uk*}converges to the optimal solution *u.* Thus, the convergence of the parameters *μk→μ* im­plies the convergence of solutions of the corresponding extremum problems. We conclude that the problem is well-posed in the sense of Hadamard.

**Conclusion 6.5.** *The proof of the well-posedness of the optimization problem in the sense of Hadamard is mostly reduced to the proof of well-posedness in the sense of Tikhonov for the fixed parameter values.*

Another conclusion is that the regularization methods used for solving ill-posed problems in the sense of Tikhonov can be applied in the practical solution of ill-posed problems in the sense of Hadamard.

 **Example**. Consider the set

*U =* { *u* ∈ *L*2(0,1)| | *u*(*t*) | ≤ 1, *t*∈(0,1) }.

and the functional



where *у* is a known square-integrable function and the system state *x* is related to the control by the formulas

.

**Problem 6*'*.** Find a control *u*∈*V* minimizing the functional *Iy* on *U.*

Problem 6*'* is well-posed, which can be established the same way as for Problem 5*'*. For any values of *у* and *z,* we have



Hence, if *у* and *z* are sufficiently close to each other, so are the corresponding values of the functional, the proximity being of the same order for every control. It follows that the assumptions of Theorem 9 hold.

**Conclusion 6.6.** *Problem 6' is* *well-posed in the* sense *of Hadamard.*

### SUMMARY

The analysis of this example yields the following conclusions.

1. The dependence of the solution on the problem parameters in op­timal control problems may be not continuous. This is caused by ill-posedness in the sense of Hadamard.
2. Optimal control problems that have a unique solution may be ill-posed in the sense of Hadamard.
3. In ill-posed problems in the sense of Hadamard, small errors in the system parameters may lead to substantial errors in the solution re­sults.
4. In ill-posed problems in the sense of Hadamard, small errors related to the algorithm procedure may lead to substantial errors in the solution results.
5. There is a relationship between well-posedness in the sense of Hadamard and well-posedness in the sense.
6. Regularization methods can be used for the practical solution of ill-posed problems.

### TASKS

Prove Tihonov’s ill-posedness for the following optimization problems:

1. 

2. 

3. 

4. 